Blow up results for a viscoelastic Kirchhoff-type equation with logarithmic nonlinearity and strong damping

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ABSTRACT. A Kirchhoff equation type with memory term competing with a logarithmic source is considered. By using potential well theory, we obtained the global existence of solution for the initial data in a stability set created from Nehari Manifold and prove blow up results for initial data in the instability set.

1. INTRODUCTION

We are investigating the following viscolelastic Kirchhoff type problem with logarithmic nonlinearity, for $(x,t) \in \Omega \times R^+$,

$$(1) \quad u_{tt} - M\left(\|\nabla u\|^2\right) \Delta u + \int_0^t g\left(t-s\right) \triangle u\left(s\right) \mathrm{d}\, s - \Delta u_t = |u|^{p-2}\, u \ln |u|,$$

$$(2) \qquad\qquad u\left(x,0\right) = u_0\left(x\right), \ u_t\left(x,0\right) = u_1\left(x\right), \ x \in \Omega,$$

$$(3) \qquad\qquad u\left(x,t\right) = 0, \ x \in \partial\Omega \times R^+,$$

where Ω is a bounded domain in $R^n, n \geq 1$ with smooth boundary $\partial\Omega$. $M\left(s\right) = \alpha + \beta s^\gamma \ (\gamma, s \geq 0)$, specially, we take $a = \beta = 1$. We impose some conditions to be specified on the kernel function $g\left(t\right)$.

This kind of wave equation (1) is called of Kirchhoff type in reason of the one-dimensional nonlinear equation (4) proposed by Kirchhoff [10] (1883),

$$(4) \qquad \frac{\partial^2 u}{\partial t^2} - \left(\frac{\tau_0}{m} + \frac{k}{2mL}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}\, x\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

where τ_0 is the initial tension, m the mass of the string and k the Young's modulus of the material of the string. This model, in connection with some problems in nonlinear elasticity, describes small vibrations of a stretched

string of the length L when only the transverse component of the tension is considered.

In [18], was proved existence and the energy decay estimate of global solutions for a extensible beam equation of Kirchhoff type with internal damping $|u_t|^{p-1}u_t$ and source term $|u|^{q-1}u$ given by

$$(5) \quad u_{tt} + \Delta^2 u + M(||\nabla u||^2)(-\Delta u) + |u_t|^{p-1}u_t = |u|^{q-1}u \quad \text{in } \Omega \times (0, T),$$

$$(6) \qquad\qquad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,$$

$$(7) \qquad\qquad u(x,t) = \frac{\partial u}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega, t \geq 0,$$

where $p \geq 1$, $q > 1$ are real numbers, Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $M(s)$ is a continuous function on $[0, +\infty)$.

The following equation
(8)

$$|u_t|^\rho u_{tt} - M\left(\|\Delta u\|^2\right)\Delta^2 u + \Delta^2 u_{tt} - \int_0^t h(t-s)\triangle^2 u(s)\,\mathrm{d}s = |u|^{\gamma-1}u\ln|u|^k$$

has been considered by Boulaaras et al. in [4]. They established polynomial decay rate estimates results of solutions. In presence of delay term with $\gamma = 1$ the equation (8) has been studied by Mezouar et al. in [17]. They proved the global existence of weak solutions and the uniform decay of the energy is derived.

In the absence of kernel function (that is $g = 0$) and by taking $M(s) = 1$ and $p = 2$, the equation (1) can be recorded in the form

$$(9) \qquad\qquad u_{tt} - \Delta u - \Delta u_t = u\ln|u|.$$

Logarithmic nonlinearity term appears frequently in partial differential equations due to their wide application in physics and other applied sciences. Problems like equation (9) is encountered naturally in quantum mechanics, inflation cosmolog, supersymmetric field theories, and a lot of different areas of physics such as, optics, geophysics and nuclear physics [5,6,11,13]. With all those specific meaning in physics, the mathematical behaviour of solution to the problem of evolution equation with such logarithmic type nonlinearity takes lots of attention by many mathematicians. We also mention some related mathematical work involving the logarithmic term in the literature see, for example, [7–9,12,15] and references therein.

When $M(s) = 1$, $g \neq 0$, then these problems turn into viscoelastic wave equation with logarithmic source term. This type problems have been handled carefully by many authors in several works [2,16,19,21,25].

In the present paper, we dedicate our study to problem (1)-(3). The structure of the work is as follows: To facilitate the description, firstly in section 2 we present the potential well. In Section 3 we present hypotheses and lemmas which will need throughout in our proof of existence of solution and

blow up in finite time. In Section 4 and section 5, respectively, we establish the global existence and blow up results.

2. Potential well

As standard, $\|\cdot\|_2$ and $\|\cdot\|_p$ denotes the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. We denote by C and C_i $(i = 1, 2, ...)$ various positive constants. We will use the Standart Lebesgue Space $L^2(\Omega)$ with the inner product and the norm. The inner product can take as

$$\langle u, v \rangle = \int u(x)v(x)\,\mathrm{d}\,x$$

and the norm is defined as

$$\|u\|_2 = \langle u, u \rangle^{\frac{1}{2}}.$$

It is well known that the energy of a PDE system is, in some sense, split into kinetic and potential energy. Following the idea of Y. Ye [24] we are able to construct a set of stability as follows. We will prove that there is a valley or a "well" of depth d created in the potential energy. If this height d is strictly positive, we find that, for solutions with initial data in the "good part" of the well, the potential energy of the solution can never escape the well. In general, it is possible for the energy from the source term to cause the blow-up in finite time.

For $u \in H_0^1(\Omega)$ we define the functional

$$(10) \qquad \mathcal{J}(\lambda u) = \frac{\lambda^2 \beta}{4(\gamma + 1)} \|u\|^{2(\gamma + 1)} - \frac{\lambda}{p} \int_\Omega u^p \ln|u|\,\mathrm{d}\,x, \quad 0 < \lambda \le 1.$$

Associated with the $\mathcal{J}$ we have the well known *Nehari Manifold* given by

$$N \overset{def}{=} \left\{ u \in H_0^1(\Omega) / 0 : \left[\frac{d}{d\lambda} J(\lambda u) \right]_{\lambda = 1} = 0 \right\}.$$

From (10) we get

$$\frac{\mathrm{d}}{\mathrm{d}\,\lambda} \mathcal{J}(\lambda u) = \frac{\lambda \beta}{2(\gamma + 1)} \|u\|^{2(\gamma + 1)} - \frac{1}{p} \int_\Omega u^p \ln|u|\,\mathrm{d}\,x,$$

then

$$\mathcal{N} \overset{def}{=} \left\{ u \in H_0^1(\Omega)/\{0\} \ : \ \frac{\lambda \beta}{2(\gamma + 1)} \|u\|^{2(\gamma + 1)} = \frac{1}{p} \int_\Omega u^p \ln|u|\,\mathrm{d}\,x \right\}.$$

We define as in the Mountain Pass theorem due to Ambrosetti and Rabinowitz [3],

$$d = \inf_{\{u \in H_0^1(\Omega)/0\}} \sup_{0 \le \lambda} J(\lambda u).$$

It is well-known that the depth of the well d is a strictly positive constant, see [[22], Theorem 4.2], and

$$d = \inf_{u \in \mathcal{N}} \mathcal{J}(u).$$

In fact, in our problem, the solution of $\dfrac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{J}(\lambda u) = 0$ is

$$\lambda_* = \left[\frac{\frac{1}{p} \int\limits_{\Omega} u^p \ln|u| \,\mathrm{d}x}{\frac{\beta}{2(\gamma+1)} \|u\|^{2(\gamma+1)}} \right].$$

We have

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \mathcal{J}(\lambda u) = \frac{\beta}{2(\gamma+1)} \|u\|^{2(\gamma+1)} > 0,$$

and then λ_* is a global minimum.

Since $\lambda_* < 1$, by straight calculation we get $\mathcal{J}(\lambda_* u) < 0$, so we introduce the sets

$$\mathcal{H}_1 = \{u \in H_0^1(\Omega); \mathcal{J}(\lambda_* u) \leq \mathcal{J}(\lambda u) \leq 0\}$$

and

$$\mathcal{H}_2 = \{u \in H_0^1(\Omega); 0 < \mathcal{J}(\lambda u)\}.$$

The potential well is defined by $\mathcal{H} = \{u \in H_0^1(\Omega) \ : \ \mathcal{J}(u) < d\} \cup \{0\}$ and partition it into two sets

$$V = \left\{ u \in \mathcal{H} \ : \ \frac{\lambda\beta}{2(\gamma+1)} \|u\|^{2(\gamma+1)} > \frac{1}{p} \int\limits_{\Omega} u^p \ln|u| \,\mathrm{d}x \right\} \cup \{0\},$$

$$W = \left\{ u \in \mathcal{H} \ : \ \frac{\lambda\beta}{2(\gamma+1)} \|u\|^{2(\gamma+1)} < \frac{1}{p} \int\limits_{\Omega} u^p \ln|u| \,\mathrm{d}x \right\}.$$

We will refer to V as the "good" part of the well and W as the "bad" part of the well. Then we define by V the set of stability for the problem (1)-(3).

3. Technical lemmas

In this section, we present hypotheses and lemmas which will need throughout this paper.

Lemma 1. *[1, 20] Let c_q be the smallest positive constant satisfying*

$$(11) \qquad\qquad \|u\|_q \leq c_q \|\nabla u\|, \quad \forall u \in H_0^1(\Omega),$$

where $2 \leq q < \infty$, if $n = 1, 2$; and $2 \leq q \leq \frac{2n}{n-2}$, if $n \geq 3$.

We state general assumptions on g :

($A1$) $g \in C^1\left([0,\infty)\right)$ is a nonincreasing and nonnegative function satisfying

$$(12) \qquad g(0) \geq 0, \ \alpha - \int_0^\infty g(s)\,ds = l > 0,$$

($A2$) There exist positive constant ϑ such that

$$(13) \qquad g'(t) \leq \vartheta g(t), t \geq 0.$$

The total energy functional $E(t)$ of the solution $u(t)$ of equation (1) is defined as follows:

$$(14) \qquad E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(\alpha - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|^2 + \frac{1}{p^2}\|u\|_p^p$$

$$+ \frac{\beta}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\int_\Omega u^p \ln|u|\,\mathrm{d}x,$$

and we have that $E(t) > 0$ in the good part of the well.

Now, we introduce the potential energy functional

$$(15) \qquad J(u) = \frac{1}{2}\left(\alpha - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|^2 + \frac{1}{p^2}\|u\|_p^p$$

$$+ \frac{\beta}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\int_\Omega u^p \ln|u|\,\mathrm{d}x,$$

and the functional

$$(16) \qquad I(u) = \left(\alpha - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|^2 + \beta\|\nabla u\|^{2(\gamma+1)}$$

$$(17) \qquad + (g \circ \nabla u)(t) - \int_\Omega u^p \ln|u|\,\mathrm{d}x,$$

for $u \in H_0^1(\Omega)$, where

$$(18) \qquad (g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(s) - \nabla u(t)\|^2\,\mathrm{d}s.$$

Then, it is easy to show that for $u \in H_0^1(\Omega)$,

$$J(u) = \frac{1}{p}I(u) + \frac{(p-2)l}{2p}\|\nabla u\|^2 + \frac{p-2\gamma-2}{2p(\gamma+1)}\beta\|\nabla u\|^{2(\gamma+1)}$$

$$(19) \qquad + \frac{(p-2)}{2p}(g \circ \nabla u)(t) + \frac{1}{p^2}\|u\|_p^p,$$

$$(20) \qquad E(t) = \frac{1}{2} \|u_t\|^2 + J(u).$$

Remark 1. From the definition of $E(t)$ with (20), $(A1) - (A2)$ and Lemma 1, we get

(21)

$$
E(t) \geq \frac{1}{2} \left(\alpha - \int_0^t g(s) \, d s \right) \|\nabla u\|^2 + \frac{1}{2} (g \circ \nabla u)(t) - \frac{1}{p} \int_\Omega u^p \ln|u| \, d x
$$

$$
\geq \frac{1}{2} \left(l \|\nabla u\|^2 + (g \circ \nabla u)(t) \right) - \frac{1}{p} \|u\|_{p+1}^{p+1}
$$

$$
\geq \frac{1}{p} \left[\frac{p}{2} \left(l \|\nabla u\|^2 + (g \circ \nabla u)(t) \right) - \|u\|_{p+1}^{p+1} \right]
$$

$$
\geq \frac{1}{p} \left[\frac{p}{2} \left(l \|\nabla u\|^2 + (g \circ \nabla u)(t) \right) - C_*^{(p+1)} l^{\frac{p+1}{2}} \|\nabla u\|_2^{p+1} \right]
$$

$$
\geq \frac{1}{p} K \left[\left(l \|\nabla u\|^2 + (g \circ \nabla u)(t) \right)^{\frac{1}{2}} \right], t \geq 0, \text{ where}
$$

$$(22) \qquad K(\sigma) = \frac{p}{2} \sigma^2 - C_*^{(p+1)} \sigma^{p+1} \text{ and } C_* = \frac{c}{\sqrt{l}}.$$

It is easy to verify that there is a maximum value of $K(\sigma)$ at

$$
\sigma_1 = \left(\frac{p}{p+1} C_*^{-(p+1)} \right)^{\frac{1}{p-1}}
$$

and the maximum value is

$$(23) \qquad E_1 = \frac{p-1}{2} \left(\frac{p}{p+1} \right)^{\frac{p+1}{p-1}} C_*^{-\frac{2(p+1)}{p-1}}.$$

Now, the following assumptions and lemmas have an important role in the proving of our main results.

Lemma 2. *Suppose that $(A1)$ and $(A2)$ hold. Then the energy functional $E(t)$ is decresing with respect to t and*
(24)
$$
E'(t) = -\frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} \left[(g' \circ \nabla u)(t) - g(t) \|\nabla u(t)\|^2 \right] \leq -\frac{1}{2} \|\nabla u_t\|^2 \leq 0,
$$

where

$$
(g' \circ \nabla u)(t) = \int_0^t g'(t-s) \int_\Omega |\nabla u(s) - \nabla u(t)|^2 \, d x \, d t.
$$

Kanıt. We multiply both sides of (1) by u_t and then integrating from 0 to t, we have

$$(25) \quad E(t) - E(0) = -\frac{1}{2} \|\nabla u_t\|^2 + \int_0^t \frac{1}{2} \left[(g' \circ \nabla u)(t) - g(t) \|\nabla u(t)\|^2 \right]$$

$$\leq -\frac{1}{2} \|\nabla u_t\|^2,$$

which yields (24) by a simple calculation. □

Lemma 3. *Let u be the solution of problem (1) with initial data satisfies $E(0) < E_1$ and $l^{\frac{1}{2}} \|\nabla u_0\| < \sigma_1$, then*

$$(26) \qquad\qquad \left(l \|\nabla u\|^2 + (g \circ \nabla u)(t) \right)^{\frac{1}{2}} < \sigma_1,$$

for $t \in [0, T)$.

Kanıt. By (22), we see that

$$K(\sigma) = \begin{cases} > 0, & 0 < \sigma < \sigma_1 \\ = 0, & \sigma < \sigma_1 \\ < 0, & \sigma > \sigma_1, \end{cases}$$

and $\lim_{\sigma \to \infty} K(\sigma) = -\infty$. We prove Lemma 3 by contradiction. Assume (26) does not satisfy. Then, by the continuity of $u(t)$ about time, there is a $t^* \in (0, T)$ to make

$$\left(l \|\nabla u\|^2 + (g \circ \nabla u)(t) \right)^{\frac{1}{2}} = \sigma_1.$$

By (21), it follows that

$$(27) \qquad\qquad E(t^*) \geq K \left[\left(l \|\nabla u\|^2 + (g \circ \nabla u)(t) \right)^{\frac{1}{2}} \right]$$

$$= K(\sigma_1)$$

$$= E_1.$$

But this is imposible by $E(t) \leq E(0) < E_1$, for $\forall t \geq 0$. Therefore, the proof is completed. □

Remark 2. From (26) and $\sigma_1 = \left(\frac{p}{p+1} C_*^{-(p+1)} \right)^{\frac{1}{p-1}} = \left(\frac{p}{p+1} \right)^{\frac{1}{p-1}} c^{-\frac{(p+1)}{p-1}} l^{\frac{2(p+1)}{p-1}}$, we obtain

$$(28) \quad l \|\nabla u\|^2 \leq l \|\nabla u\|^2 + (g \circ \nabla u)(t) < \sigma_1^2 = \left(\frac{p}{p+1} \right)^{\frac{2}{p-1}} l^{\frac{4(p+1)}{p-1}} c^{-\frac{2(p+1)}{p-1}},$$

which implies that

(29)

$$
\begin{aligned}
I(u) &= \left(\alpha - \int_0^t g(s)\, \mathrm{d}s \right) \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + (g \circ \nabla u)(t) - \int_\Omega u^p \ln |u|\, \mathrm{d}x \\
&\geq \left(\alpha - \int_0^\infty g(s)\, \mathrm{d}\, ds \right) \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + (g \circ \nabla u)(t) - \int_\Omega u^p \ln |u|\, \mathrm{d}x \\
&\geq l \|\nabla u\|^2 - \|u\|_{p+1}^{p+1} \\
&\geq l \|\nabla u\|^2 - c^{p+1} \|\nabla u\|_2^{p+1} \geq 0.
\end{aligned}
$$

Further, by (19), we get

$$
(30) \qquad
\begin{aligned}
J(u) &= \frac{1}{p} I(u) + \frac{(p-2)l}{2p} \|\nabla u\|^2 + \frac{p - 2\gamma - 2}{2p(\gamma+1)} \beta \|\nabla u\|^{2(\gamma+1)} \\
&\quad + \frac{p-2}{2p} (g \circ \nabla u)(t) + \frac{1}{p^2} \|u\|_p^p \\
&\geq \frac{p-2}{2p} \left[l \|\nabla u\|^2 + (g \circ \nabla u)(t) \right] + \frac{1}{p} I(u).
\end{aligned}
$$

By (29), (30), (20) and Lemma 2 we conclude that

$$
(31) \qquad l \|\nabla u\|^2 \leq \frac{2p}{p-2} J(u) \leq \frac{2p}{p-2} E(t) \leq \frac{2p}{p-2} E(0) \leq \frac{2p}{p-2} E_1.
$$

Since $\log x < x$ for any $x > 0$, using Lemma 1, we get that

$$
(32) \qquad \int_\Omega u^p \ln |u|\, \mathrm{d}x \leq \|u\|_{p+1}^{p+1}
$$

$$
(33) \qquad
\begin{aligned}
&\leq c^{p+1} \|\nabla u\|^{p+1} \\
&\leq \frac{c^{p+1}}{l} \left(\frac{2p}{l(p-2)} E(0) \right)^{\frac{p-1}{2}} l \|\nabla u\|^2 \\
&= \mu l \|\nabla u\|^2 \\
&\leq \frac{2p\mu}{p-2} E(t),
\end{aligned}
$$

where $\mu = \frac{c^{p+1}}{l} \left(\frac{2p}{l(p-2)} E(0) \right)^{\frac{p-1}{2}}$. Notice that $E(0) < E_1$ if and only if

$$
(34) \qquad \mu = \frac{c^{p+1}}{l} \left(\frac{2p}{l(p-2)} E(0) \right)^{\frac{p-1}{2}} < 1.
$$

4. Global existence

We start this section, present the local existence theorem. The proof of theorem 1 can be found similar to [14, 23].

Theorem 1. *Let $u_0 \in W$, $u_1 \in H_0^1(\Omega)$. Assume that $(A1)$, $E(0) < E_1$ and $l\|\nabla u\|^2 \le \sigma_1^2$ hold. Then problem (1) has a unique solution $u(x,t)$ satisfies $u(t) \in L^\infty\left(0, T; H_0^1(\Omega)\right)$ with $u_t(t) \in L^\infty\left(0, T; H_0^1(\Omega)\right)$.*

Moreover, at least one of the following statements holds true, $T = \infty$ or $\|u_t\|^2 + \|\Delta u\|^2 \to \infty$ as $t \to T^-$.

Now we are in position to state and prove the global existence result.

Theorem 2. *Let $u_0 \in W$, $u_1 \in H_0^1(\Omega)$. Suppose that $(A1)$, $0 < E(0) < E_1$ and $l\|\nabla u\|^2 \le \sigma_1^2$ hold. Then problem (1)-(3) admits a global weak solution $u(t) \in L^\infty\left(0, \infty; H_0^1(\Omega)\right)$ with $u_t(t) \in L^\infty\left(0, \infty; H_0^1(\Omega)\right)$.*

Kanıt. Our aim is to show that the solutions exist when $T = \infty$. First, multiplying (1) by $-2\Delta u$, and integrating it over Ω, we obtain

(35)
$$\frac{d}{dt}\left\{\|\Delta u\|^2 - 2\int_\Omega u_t \Delta u \, dx\right\} + 2M\left(\|\nabla u\|^2\right)\|\Delta u\|^2$$

$$\le 2\|\nabla u_t\|^2 - 2\int_\Omega |u|^{p-2} u \ln|u| \Delta u \, dx + 2\int_0^t g(t-s)\int_\Omega \triangle u(s) \triangle u(t) \, dx \, ds.$$

We now estimate the last term in the right side of (35) as follows

(36)
$$2\int_0^t g(t-s)\int_\Omega \triangle u(s) \triangle u(t) \, dx \, ds \le 2\varepsilon \|\Delta u\|^2 + \frac{\|g\|_{L^1}}{2\varepsilon}\int_0^t g(t-s)\|\triangle u(s)\|^2 \, ds.$$

where $0 < \varepsilon < \frac{\|g\|_{L^1}}{2}$.

Thus, by inserting (36) in (35) and then multiplying it by η, $0 < \eta \le 1$, we have

(37)
$$\frac{d}{dt}\left\{\eta\|\Delta u\|^2 - 2\eta\int_\Omega u_t \Delta u \, dx\right\} + 2\eta\left(M\left(\|\nabla u\|^2\right) - \varepsilon\right)\|\Delta u\|^2$$

$$\le 2\eta\|\nabla u_t\|^2 - 2\eta\int_\Omega |u|^{p-2} u \ln|u| \Delta u \, dx + \eta\frac{\|g\|_{L^1}}{2\varepsilon}\int_0^t g(t-s)\|\triangle u(s)\|^2 \, ds.$$

Then adding $2E'(t)$ to (37), we have

(38) $\quad \dfrac{\mathrm{d}}{\mathrm{d}\,t} E^*(t) + 2\eta \left(M\left(\|\nabla u\|^2\right) - \varepsilon \right) \|\Delta u\|^2 + 2(1-\eta)\|\nabla u_t\|^2$

$$\leq -2\eta \int_\Omega |u|^{p-2}\, u \ln|u|\, \Delta u \, \mathrm{d}\,x + \eta \frac{\|g\|_{L^1}}{2\varepsilon} \int_0^t g(t-s)\|\triangle u(s)\|^2 \, \mathrm{d}\,s,$$

where

(39) $$E^*(t) = 2E(t) + \eta \|\Delta u\|^2 - 2\eta \int_\Omega u_t \Delta u \, \mathrm{d}\,x.$$

Applying Young's inequality, we obtain

(40) $$\left| 2\eta \int_\Omega u_t \Delta u \, \mathrm{d}\,x \right| \leq 2\eta \|u_t\|^2 + \frac{\eta}{2}\|\Delta u\|^2.$$

By Remark 2 and (30) we see clearly that $J(t) \geq 0$, then by (20) we get

(41) $$\|u_t\|^2 \leq 2E(t).$$

By using of (40) and (41), (39) becomes

(42) $$E^*(t) \geq (1-2\eta)\|u_t\|^2 + \frac{\eta}{2}\|\Delta u\|^2$$

$$= \frac{1}{5}\left(\|u_t\|^2 + \|\Delta u\|^2 \right), \text{ where } \eta = \frac{2}{5}.$$

Moreover, we notice that

$$2\left| \int_\Omega |u|^{p-2} \ln|u|\, u \Delta u \, \mathrm{d}\,x \right| \leq \int_\Omega |u|^{p-2}\ln|u|\,|\nabla u|^2 \, \mathrm{d}\,x$$

$$\leq \int_\Omega |u|^{p-1}\,|\nabla u|^2 \, \mathrm{d}\,x$$

$$\leq \|u\|_{(p-1)\theta_1}^{p-1}\,\|\nabla u\|_{2\theta_2}^2,$$

where $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$, so that, we put $\theta_1 = 1$ and $\theta_2 = \infty$, if $n = 1$; $\theta_1 = 1 + \varepsilon_1$, if $n = 2$; and $\theta_1 = \frac{n}{2}$, $\theta_2 = \frac{n}{n-2}$, if $n \geq 3$. Then, by using Lemma 1, we have

$$2\left| \int_\Omega |u|^{p-2}\ln|u|\, u\Delta u \, \mathrm{d}\,x \right| \leq c_s^p \|\nabla u\|_2^{p-1}\|\Delta u\|^2.$$

By (31) and (42), we obtain

(43) $$2\left| \int_\Omega |u|^{p-2}\ln|u|\, u\Delta u \, \mathrm{d}\,x \right| \leq c_1 E^*(t),$$

where $c_1 = 5c_s^p \left(\frac{2p}{l(p-2)} E(0) \right)^{\frac{p-1}{2}}$.

Substituting (43) into (38), and then integrating it over $(0, t)$, we get

$$E^*(t) + \frac{4}{5} \left(\alpha - \varepsilon - \frac{\|g\|_{L^1}^2}{4\varepsilon} \right) \int_0^t \|\triangle u(s)\|^2 \, \mathrm{d}s$$

$$\leq E^*(0) + \int_0^t c_1 E^*(s) \, ds,$$

hence by choosing $\varepsilon = \frac{\|g\|_{L^1}}{2}$ we see that

$$E^*(t) \leq E^*(0) + \int_0^t c_1 E^*(s) \, \mathrm{d}s.$$

Then, by using of Gronwall's inequality, we arrive at

$$E^*(t) \leq E^*(0) \, e^{c_1 t},$$

for any $t \geq 0$. Therefore by (42) and Theorem 1, we obtain $T = \infty$. $\qquad \square$

5. Blow up

In this section, we prove the blow up result of solution for the problem (1)-(3), taking into account that the initial data is localized in instability set.

Theorem 3. *Suppose that $u_0 \in W$ and $u_1 \in H_0^1(\Omega)$ and Lemma 2 holds. Then the solution of the equation* (1) *blows up as time t goes to infinity as long as $E(0) < 0$ and $I(u_0) > 0$.*

Kanıt. Our aim is to show that the finite time blow up of solution to equation (1). If it is not case, we suppose that the solution $u(x, t)$ is global. For any $T_0 > 0$, defining the following auxiliary function

$$G(t) = \|u\|^2 + \int_0^t \|\nabla u(\tau)\|^2 \, \mathrm{d}\tau + (T_0 - t) \|\nabla u_0\|^2,$$

then we see clear that for $G(t) > 0$ for all $t \in [0, T_0]$. It is obvious that $G(t) > 0$. In view of the continuity of $G(t)$ in t, we obtain that there is a $\rho > 0$ such that

$$G(t) \geq \rho,$$

where ρ is independent of T_0.

Then by $t \in [0, T_0]$, we have

(44) $$G'(t) = 2(u, u_t) + \|\nabla u\|^2 - \|\nabla u_0\|^2$$

$$= 2 \left(u, u_t \right) + \int_0^t \left(\nabla u \left(\tau \right), \nabla u_\tau \left(\tau \right) \right) \mathrm{d}\, \tau$$

and

$$(45) \qquad G'' \left(t \right) = 2 \left\| u_t \right\|^2 + 2 \int_\Omega u_{tt} u \, \mathrm{d}\, x + 2 \left(\nabla u, \nabla u_t \right)$$

$$= 2 \left\| u_t \right\|^2 - 2 \int_\Omega M \left(\left\| \nabla u \right\|^2 \right) \left\| \nabla u \right\|^2 \mathrm{d}\, x$$

$$+ 2 \int_0^t g \left(t - s \right) \int_\Omega \nabla u \left(s \right) \nabla u \left(t \right) \mathrm{d}\, s \, \mathrm{d}\, x + 2 \int_\Omega u^p \ln |u|$$

$$= 2 \left\| u_t \right\|^2 - \left(\alpha - \int_0^t g \left(s \right) \mathrm{d}\, s \right) \left\| \nabla u \right\|^2 - 2 \left\| \nabla u \right\|^{2(\gamma+1)}$$

$$+ \int_0^t g \left(t - s \right) \left\| \nabla u \left(s \right) \right\|^2 \mathrm{d}\, s - \left(g \circ \nabla u \right) \left(t \right) + 2 \int_\Omega u^p \ln |u|$$

$$\geq 2 \left\| u_t \right\|^2 - 2 I \left(u \right).$$

By (44), we see that

$$(46) \qquad \left(G' \left(t \right) \right)^2 = 4 \left[\left(u, u_t \right)^2 + \left(\int_0^t \left(\nabla u \left(\tau \right), \nabla u_\tau \left(\tau \right) \right) \mathrm{d}\, \tau \right)^2 \right]$$

$$+ 4 \left(u, u_t \right) \int_0^t \left(\nabla u \left(\tau \right), \nabla u_\tau \left(\tau \right) \right) \mathrm{d}\, \tau.$$

Now we shall estimate the each terms in (46) by Cauchy-Schwarz, Hölder and Young's inequalities as follows

$$(47) \qquad\qquad\qquad\qquad \left(u, u_t \right)^2 \leq \left\| u \right\|^2 \left\| u_t \right\|^2,$$

$$(48) \qquad \left(\int_0^t \left(\nabla u \left(\tau \right), \nabla u_\tau \left(\tau \right) \right) \mathrm{d}\, \tau \right)^2 \leq \int_0^t \left\| \nabla u \right\|^2 \mathrm{d}\, \tau \int_0^t \left\| \nabla u_\tau \right\|^2 \mathrm{d}\, \tau.$$

By combining (47) and (48) and using Young's inequality, we have

$$(49) \qquad 2 \left(u, u_t \right) \int_0^t \left(\nabla u \left(\tau \right), \nabla u_\tau \left(\tau \right) \right) \mathrm{d}\, \tau$$

$$\leq 2 \left\| u \right\| \left\| u_t \right\| \left(\int_0^t \left\| \nabla u \right\|^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \left\| \nabla u_\tau \right\|^2 d\tau \right)^{\frac{1}{2}}$$

$$\leq \left\| u \right\|^2 \int_0^t \left\| \nabla u_\tau \right\|^2 d\tau + \left\| u_t \right\|^2 \int_0^t \left\| \nabla u \right\|^2 d\,d\tau.$$

Inserting (47)-(49) into (46) becomes

$$(50) \qquad \left(G'(t) \right)^2 \leq 4 \left\| u \right\|^2 \left\| u_t \right\|^2 + 4 \int_0^t \left\| \nabla u \right\|^2 d\tau \int_0^t \left\| \nabla u_\tau \right\|^2 d\,d\tau$$

$$+ 4 \left(\left\| u \right\|^2 \int_0^t \left\| \nabla u_\tau \right\|^2 d\tau + \left\| u_t \right\|^2 \int_0^t \left\| \nabla u \right\|^2 d\tau \right)$$

$$= 4 \left\| u \right\|^2 \left(\left\| u_t \right\|^2 + \int_0^t \left\| \nabla u_\tau \right\|^2 d\tau \right)$$

$$+ 4 \int_0^t \left\| \nabla u \right\|^2 d\,d\tau \left(\left\| u_t \right\|^2 + \int_0^t \left\| \nabla u_\tau \right\|^2 d\tau \right)$$

$$= 4 \left(\left\| u \right\|^2 + \int_0^t \left\| \nabla u \right\|^2 d\tau \right) \left(\left\| u_t \right\|^2 + \int_0^t \left\| \nabla u_\tau \right\|^2 d\tau \right)$$

$$\leq 4 G(t) \left(\left\| u_t \right\|^2 + \int_0^t \left\| \nabla u_\tau \right\|^2 d\tau \right).$$

Then by (45) and (50) we deduce

$$(51) \qquad G''(t) G(t) - \frac{p+2}{4} \left(G'(t) \right)^2$$

$$\geq G(t) \left[G''(t) - (p+2) \left(\left\| u_t \right\|^2 + \int_0^t \left\| \nabla u_\tau \right\|^2 d\tau \right) \right]$$

$$\geq G(t) \left[2 \left\| u_t \right\|^2 - 2I(u) - (p+2) \left(\left\| u_t \right\|^2 + \int_0^t \left\| \nabla u_\tau \right\|^2 d\tau \right) \right]$$

$$= G(t) \, \xi(t),$$

where

$$(52) \qquad \xi(t) = -p\|u_t\|^2 - 2I(u) - (p+2)\int_0^t \|\nabla u_\tau\|^2 \, \mathrm{d}\tau.$$

By Lemma 2 and (19) we obtain

$$(53) \quad E(0) \geq \int_0^t \|\nabla u_\tau\|^2 \, \mathrm{d}\tau + E(t)$$

$$= \int_0^t \|\nabla u_\tau\|^2 \, \mathrm{d}\tau + \frac{1}{2}\|u_t\|^2 + J(u)$$

$$= \int_0^t \|\nabla u_\tau\|^2 \, \mathrm{d}\tau + \frac{1}{2}\|u_t\|^2 + \frac{1}{p}I(u) + \frac{(p-2)\,l}{2p}\|\nabla u\|^2$$

$$+ \frac{p-2\gamma-2}{2p\,(\gamma+1)}\beta\|\nabla u\|^{2(\gamma+1)} + \frac{(p-2)}{2p}(g\circ\nabla u)(t) + \frac{1}{p^2}\|u\|_p^p.$$

Inserting (53) into (52) we obtain

$$(54) \quad \xi(t) > (p-2)\int_0^t \|\nabla u_\tau\|^2 \, \mathrm{d}\tau - 2pE(0) + (p-2)\,l\,\|\nabla u\|^2$$

$$+ \frac{p-2\gamma-2}{(\gamma+1)}\beta\|\nabla u\|^{2(\gamma+1)} + (p-2)(g\circ\nabla u)(t) + \frac{1}{p^2}\|u\|_p^p.$$

Under the condition $E(0) < 0$, we obtain

$$(55) \qquad (p-2)\,l\,\|\nabla u\|^2 - 2pE(0) > 0.$$

Hence by (55) we arrive at that

$$(56) \qquad \xi(t) > (p-2)\left(\int_0^t \|\nabla u_\tau\|^2 \, \mathrm{d}\tau + (g\circ\nabla u)(t)\right)$$

$$+ \frac{p-2\gamma-2}{(\gamma+1)}\beta\|\nabla u\|^{2(\gamma+1)} + \frac{1}{p^2}\|u\|_p^p$$

$$> \vartheta > 0.$$

Then, by using (56), we can see clearly that

$$G''(t)\,G(t) - \frac{p+2}{4}\left(G'(t)\right)^2 \geq G(t)\,\vartheta.$$

Let $y(t) = G(t)^{\frac{-p+2}{4}}$, then we obtain

$$y''(t) \leq -\frac{p-2}{4}\vartheta y(t)^{\frac{p+2}{p-2}}, \quad t \in [0, T_0].$$

That is

$$\lim_{t \to T^*} y(t) = 0,$$

where T^* is independent of initial choice of T_0 and $T^* < T_0$. Therefore we can conclude that

$$\lim_{t \to T^*} G(t) = 0. \qquad \qquad \square$$

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